AD-A224 884

Technical Document 1827
May 1990

Orderings of N-Tuples

R. F. Freund

NAVAL OCEAN SYSTEMS CENTER
San Diego, California 92152-5000

J. D. FONTANA, CAPT, USN
Commander

R. M. HILLYER
Technical Director

ADMINISTRATIVE INFORMATION

This work was performed by the Ashore Command Centers Branch, Code 423, Naval Ocean Systems Center, under an in-house block program.

Released by
R. E. Pierson, Head
Ashore Command
Centers Branch

Under authority of
J. A. Salzmann, Jr., Head
Intelligence Centers
Division

JG

CONTENTS

SPECIAL CASE

PROOF

We begin with the special case of n-tuples which sum to a given value, k, and build up to the general case. Let Z^+ denote the positive integers and let $S(n, k) = \{\alpha = (a_1, \ldots, a_n), a_i \in Z^+ s.t. \sum a_1 = k\}$ ordered lexicographically. It is well known that $|S(n, k)|$ is simply the binomial coefficient $(k - l)$ choose $(n - l)$, i.e.,

$$|S(n,k)| = \binom{k-1}{n-1} . \tag{1}$$

Now given an $\alpha \in S(n, k)$, can we calculate its position in the ordering? Certainly $(1, 1, \ldots, k + 1 - n)$ is first and $(k + 1 - n, 1, \ldots, 1)$ is last. Let the order function on $S(n, k)$ be $f_{n,k}: S(n, k) \rightarrow \{1, 2, \ldots, |S(n,k)|\}$.

LEMMA 1.1

Let $S(n, k)$ be the set of n-tuples of positive integers which sum to k, ordered lexicographically. Let $\alpha = (a_1, \ldots, a_n)$ and define $\sigma_j(\alpha) = \sum_{i=j}^{n} a_i$. Then the position of α is given by

$$f_{n,k}(a) = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1} + (a) - 1}{n-j} . \tag{2}$$

PROOF

Consider a fixed $\alpha = (a_1, \ldots, a_n)$ in $S(n, k)$. Let $\beta = b_1, \ldots, b_n)$, $T_j = \{\beta \in S(n, k) | b_1 = a_1, \ldots, b_{j-1} = a_{j-1}, b_j > a_j\}$ and let $T = U T_j$, $j = 1, \ldots$ n. Then certainly $T = \{\beta \in S(n, k) | \beta > \alpha\}$ and $T_i \cap T_j = \phi$ if $i \neq j$, so we have $|T| = \sum_{1 \leq j < n} |T_j|$ (since $T_n = \phi$). But $\{(T_j = (a_1, \ldots, a_{j-1}, a_j + g_1, g_2, \ldots g_{n-j+1}) | g = (g_1, \ldots g_{n-j+1}) \in S(n - j + 1, k - a_1 - \ldots - a_j)\}$. Hence,

$$|T_j| = \binom{k - a_1 - \ldots - a_j - 1}{n-j} = \binom{\sigma_{j+1} + (a) - 1}{n-j} . \tag{3}$$

Therefore, we can conclude that $f_{n,k}(\alpha) = |S(n, k)| - |\{b \in S(n, k) \; s.t. \; \beta > \alpha\}|$

$$= \binom{k-1}{n-1} - |T| = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} |T_j| = \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1} + (a) - 1}{n-j} .$$

qed

N-TUPLES OF POSITIVE INTEGERS

Now let $Z^n = Z^+ \times \ldots \times Z^+$ be the set of all n-tuples of positive integers. Note that $Z^n = U_{n \le k}$. Let Z^n be ordered by $S(n, n) < S(n, n+1) < S(n, n+2) < \ldots$ where the $S(n, k)$ are ordered lexicographically as before. Let $f_n : Z^n \rightarrow Z$ be the order function for this space.

THEOREM 1.1

Let Z^n be the set of all n-tuples of positive integers ordered as above. Let $\alpha = (a_1, \ldots, a_n) \in Z^n$ and define $\sigma_j(\alpha) = \sum_{i=j}^{n} a_i$. The the position of α is given by

$$f_n(a) = \binom{\sigma_j(a)}{n} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a) - 1}{n-j}. \tag{4}$$

PROOF

By ordering on Z^n and the lemma we have

$$f_n(a) = \sum_{j=1}^{\sigma_1(a)-1} |S(n,k)| + f_{n,\sigma_1(a)}(a) = \sum_{j=1}^{\sigma_1(a)-1} \binom{j-1}{n-1} + f_{n,\sigma_1(a)}(a)$$

$$= \binom{\sigma_1(a)-1}{n} + f_{n,\sigma_1(a)}(a) = \binom{\sigma_1(a)-1}{n} + \binom{\sigma_1(a)-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}$$

$$= \binom{\sigma_1(a)}{n} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} \quad \left[\begin{array}{c} \text{using the basic combinatorial identity} \\ \binom{a+1}{b} = \binom{a}{b} + \binom{a}{b} \end{array} \right].$$

qed

COROLLARIES

Next I give two corollaries of lemma 1.1.

COROLLARY 3.1

Let $\mathbf{S}(k) = \mathbf{U}_{1 \le n \le k}\ \mathbf{S}(n,k)$ ordered by $\mathbf{S}(1,k) < \ldots \mathbf{S}(k,k)$ where the $\mathbf{S}(n,k)$ are ordered lexicographically as before. Then the lexicographic order function, $f^{(k)}$, of $\alpha = (a_1, \ldots, a_n) \in \mathbf{S}(k)$ is given by

$$f^{(k)}(a) = \sum_{\lambda=1}^{n} \binom{k-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}. \tag{5}$$

PROOF

Using lemma 1.1, we have

$$f^{(k)}(a) = \sum_{\lambda=1}^{n-1} |\mathbf{S}(n,k)| + f_{n,k}(a) = \sum_{\lambda=1}^{n-1} \binom{k-1}{\lambda-1} + \binom{k-1}{n-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}$$

$$= \sum_{\lambda=1}^{n} \binom{k-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}$$

qed

COROLLARY 2

Let $\mathbf{Z}^* = \mathbf{U}_{1 \le n < \infty}\ \mathbf{Z}^n = \{\alpha = (a_1, \ldots, a_n),\ a_i \in \mathbf{Z}^+\}$ ordered by $\mathbf{S}(1) < \mathbf{S}(2) < \ldots$ with the $\mathbf{S}(\mathrm{k})$ defined and ordered as above. Then the order of function, f^*, of $\alpha = (a_1, \ldots, a_n) \in \mathbf{Z}^*$ is given by

$$f^*(a) = 2^{\sigma_1(a)-1} + \sum_{\lambda=1}^{n} \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}. \tag{6}$$

PROOF

Since $|S(m)| = \sum_{j=1}^{m} |S(j, m)| = \sum_{j=1}^{m} \binom{m-1}{j-1} = 2^{(m-1)}$, we have, by the ordering on Z^*,

$$f^*(a) = \sum_{\lambda=1}^{\sigma_1(a)-1} |S(m)| + f^{\sigma_1(a)-1}(a) = \sum_{m=1}^{\sigma_1(a)-1} 2^{m-1} + \sum_{\lambda=1}^{n} \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j}$$

$$= 2^{\sigma_1(a)-1} + \sum_{\lambda=1}^{n} \binom{\sigma_1(a)-1}{\lambda-1} - \sum_{j=1}^{n-1} \binom{\sigma_{j+1}(a)-1}{n-j} .$$

qed

INVERSE FUNCTION

Let us consider the inverse function, $f_n^{-1}: Z \to Z^n$. There does not appear to be a closed form solution, but it is readily computable. Using this, one can easily implement an algorithm to calculate $\phi_{k,n}: Z^k \to Z^n$ by $\phi_{k,n} = f_n^{-1} f_k$. Now suppose p in Z^+. To compute $f_n^{-1}(p) = (a_1, \ldots, a_n)$ first find the smallest k_1 *s.t.* $\binom{k_1}{n} \geq p$. Then we find successively largest k_1, i $= 2, \ldots, n$, *s.t.* $\binom{k_1}{n} - \sum_{j=1}^{i-1} \binom{k_{j+1}}{n-j} \geq p$. For the n^{th} case, this expression will be an equality. Then $k_i - \sigma_1(f_n^{-1}(p))$ for $i = 1, \ldots, n$ and given all the σs, one easily finds $(a_1, \ldots, a_n)$.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1 AGENCY USE ONLY (Leave blank)

2. REPORT DATE
May 1990

3. REPORT TYPE AND DATES COVERED

4 TITLE AND SUBTITLE
ORDERINGS OF N-TUPLES

5 FUNDING NUMBERS
PE: 0602234N
RS34C77
WU: DN300 086

6 AUTHOR(S)
R. F. Freund

7 PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Ocean Systems Center
San Diego, CA 92152-5000

8. PERFORMING ORGANIZATION REPORT NUMBER
NOSC TD 1827

9 SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Naval Ocean Systems Center
San Diego, CA 92152-5000

10 SPONSORING/MONITORING AGENCY REPORT NUMBER

11 SUPPLEMENTARY NOTES

12a DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13 ABSTRACT (Maximum 200 words)

The author defines several orderings on n-tuples of positive integers and shows how to compute the position of a given n-tuple with respect to these orderings. In addition, an algorithm is shown to compute the inverse function.

14 SUBJECT TERMS

15. NUMBER OF PAGES
9

16 PRICE CODE

17 SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19 SECURITY CLASSIFICATION OF ABSTRACT	20 LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAME AS REPORT